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A Characterization of Norm Continuity of Propagators for Second Order Abstract Differential Equations

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Abstract—In this paper, we obtain a concise characterization of norm continuity for $t > 0$ of propagators for the complete second order abstract differential equation on a Banach space E,

$$u''(t) + Bu'(t) + Au(t) = 0, \qquad t \geq 0,$$

where $B \in L(E)$. As a consequence, we discover that a strongly continuous cosine operator function or operator group is norm continuous for $t > 0$ if and only if its generator is bounded.

Keywords—Characterization, Norm continuity, Propagator, Second order abstract differential equation.

1. INTRODUCTION

Since the work of Lions [1] came out in 1957, a great deal of studies of the complete second order differential equation

$$u''(t) + Bu'(t) + Au(t) = 0, \qquad t \geq 0, \tag{1.1}$$

where A, B are linear operators on an abstract space E, have been done by many researchers. For references see, e.g., [2–6]; some of the more recent literatures are, e.g., [7–18] and one can find further references therein. Following these works, the present note is devoted to exploration of the characterization of norm continuity for $t > 0$ of propagators for (1.1) in case when B is bounded and E is a Banach space.

Throughout this paper, we assume that E is a Banach space with norm $\|\cdot\|$ and the Cauchy problem for (1.1) is well posed. By $L(E)$, we denote the set of all bounded linear operators from E

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TeX to E. For any operator A, $D(A)$ stands for the domain, $R(A)$ the range, $\rho(A)$ the resolvent set of A and $\sigma(A)$ the spectrum of A. Finally, $\mathbf{C}$ will denote the set of complex numbers.

Recall that the Cauchy problem for (1.1) is said to be well posed in $[0, \infty)$ if

(a) there exist dense subspaces D_0, D_1 of E such that if $u_0 \in D_0$, $u_1 \in D_1$, then (1.1) has a solution $u(\cdot)$ with $u(0) = u_0$, $u'(0) = u_1$;

(b) there exists a nondecreasing, nonnegative function $N(t)$ defined in $t \geq 0$ such that

$$\|u(t)\| \leq N(t)\,(\|u(0)\| + \|u'(0)\|), \qquad t \geq 0 \tag{1.2}$$

for any solution of (1.1).

For $t \geq 0$, $u_0 \in D_0$, $u_1 \in D_1$, define

$$C(t)u = u(t), \qquad S(t)v = v(t),$$

where $u(t)$ and $v(t)$ are the solutions of (1.1) with $\binom{u(0)}{u'(0)} = \binom{u}{0}$ and $\binom{v(0)}{v'(0)} = \binom{0}{v}$, respectively. According to (1.2), $C(t)$ and $S(t)$ are bounded operators on D_0 and D_1, respectively. Since $\overline{D}_0 = E$, $\overline{D}_1 = E$, we can extend $C(t)$, $S(t)$ to all of E as bounded operators, which we denote by the same symbols. $C(t)$, $S(t)$ are called the propagators of (1.1).

If $B = 0$ in (1.1), the propagator $C(t)$ of (1.1) is just the strongly continuous cosine operator function on E (see, e.g., [3,4]).

In this paper, we prove (Theorem 2.1 below) that in the case of $B \in L(E)$, $C(t)$, or $S'(t)$ of (1.1) is norm continuous for $t > 0$ if and only if A is bounded; in the otherwise case, a simple and clear example in Section 4 shows that even if both of $C(t)$ and $S'(t)$ are norm continuous for $t > 0$, A could be unbounded.

As a consequence of Theorem 2.1, we find that for a strongly continuous cosine operator function or operator group, it is norm continuous for $t > 0$ if and only if its generator is bounded. It is interesting to compare this with the case of strongly continuous operator semigroups. It is known that many unbounded operators generate strongly continuous semigroups which are norm continuous for $t > 0$, such as analytic semigroups (see, e.g., [2,4,19]); on the other hand, no unbounded operator generates a strongly continuous semigroup which is norm continuous at $t = 0$. This also indicates that for general operator families, the norm continuity for $t > 0$ does not imply the norm continuity at $t = 0$.

We note that it has been shown that for a cosine operator function $C(t)$, the norm continuity for $t \in (-\infty, \infty)$ implies the boundedness of the generator A. But the previous proofs depend heavily on the norm continuity at $t = 0$ of $C(t)$ (see, e.g., [20,21]). It can be seen that, Theorem 2.1 (even Corollary 2.3) here is a nontrivial generalization of this result; and the approach used, which is totally different from that in [20] or [21] and reveals the boundedness of A without the assumption of norm continuity at $t = 0$ (of $C(t)$), is concise and nontrivial.

2. RESULTS

THEOREM 2.1. *Let $B \in L(E)$. Then $C(t)$ or $S'(t)$ is norm continuous for $t > 0$ if and only if $A \in L(E)$.*

COROLLARY 2.2. *Let $B \in L(E)$. Then both propagators of (1.1) are norm continuous for $t > 0$ if and only if $A \in L(E)$.*

An $L(E)$-valued function $C(t)$ defined in $(-\infty, \infty)$ is called a strongly continuous cosine operator function on E if it satisfies (see, e.g., [3,4,20,21])

(i) $C(0) = I$ and $2C(s)C(t) = C(s+t) + C(s-t)$, for $s, t \in (-\infty, \infty)$;

(ii) $C(\cdot)u : (-\infty, \infty) \to E$ is continuous for $u \in E$.

The generator A of $C(t)$ is defined by

$$D(A) = \left\{ u \in E;\ \lim_{t\to 0+} \frac{2}{t^2}(C(t)u - u) \text{ exists} \right\},$$
$$Au = \lim_{t\to 0+} \frac{2}{t^2}(C(t)u - u), \qquad \text{for } u \in D(A).$$

COROLLARY 2.3. *Let $\{C(t)\}_{t\in(-\infty,\ \infty)}$ be a strongly continuous cosine operator function on E. Then the following statements are equivalent.*

(i) *$C(t)$ is norm continuous for $t > 0$.*
(ii) *$C(t)$ is norm continuous at $t = 0$.*
(iii) *$C(t)$ is norm continuous for $t \in (-\infty, \infty)$.*
(iv) *The generator A of $C(t)$ is bounded.*

COROLLARY 2.4. *Let $\{T(t)\}_{t\in(-\infty,\ \infty)}$ be a strongly continuous operator group on E. Then the following statements are equivalent.*

(i) *$T(t)$ is norm continuous for $t > 0$.*
(ii) *$T(t)$ is norm continuous at $t = 0$.*
(iii) *$T(t)$ is norm continuous for $t \in (-\infty, \infty)$.*
(iv) *The generator G of $T(t)$ is bounded.*

3. PROOFS

PROOF OF THEOREM 2.1. By virtue of [22, Theorem 4.1, p. 603], we have

(i) $S(t)u$ is continuously differentiable in $t \geq 0$, for all $u \in E$;
(ii) $BS(t)u$ is continuous in $t \geq 0$, for all $u \in E$;

i.e., the Cauchy Problem for (1.1) is strongly wellposed [13, p. 177]. It follows from [3, Theorem 3.2, p. 277, and Corollary 3.7, p. 287] that there exist constants C, $\omega > 0$ such that for $t \geq 0$,

$$\|S(t)\|,\ \|C(t)\|,\ \|S'(t)\| \leq Ce^{\omega t}, \tag{3.1}$$

and for $\operatorname{Re}\lambda > \omega$, $\Delta^{-1}(\lambda) := (\lambda^2 + \lambda B + A)^{-1} \in L(E)$ and

$$\Delta^{-1}(\lambda)u = \int_0^\infty e^{-\lambda t} S(t)u\, dt, \qquad u \in E.$$

Therefore, for $\operatorname{Re}\lambda > \omega$, $u \in E$,

$$\lambda\Delta^{-1}(\lambda)u = \int_0^\infty e^{-\lambda t} S'(t)u\, dt. \tag{3.2}$$

By formula (3.12) in [3, p. 279], we have that for each $u \in D(A) \bigcap D(B) = D(A)$

$$S'(t)u = C(t)u - S(t)Bu, \qquad t \geq 0. \tag{3.3}$$

By the boundedness of B and the denseness of $D(A)$, (3.3) holds for each $u \in E$. Therefore,

$$S(t)u = \int_0^t [C(t) - S(t)B]u\, dt, \qquad t \geq 0, \quad u \in E, \tag{3.4}$$

which implies that $S(t)$ is norm continuous in $t \geq 0$. Thus, we see by (3.3) that $S'(t)$ is norm continuous for $t > 0$ if and only if $C(t)$ is norm continuous for $t > 0$.

NECESSITY. Let $S'(t)$ be norm continuous in $t > 0$. For every complex number λ with $\operatorname{Re}\lambda > \omega$, define $F : \mathbf{C} \times (-\infty, \infty) \to L(E)$ as follows

$$F(\lambda, t) = \begin{cases} e^{-\operatorname{Re}\lambda t} S'(t), & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \tag{3.5}$$

By (3.2), for $\operatorname{Re}\lambda > \omega$, $\operatorname{Im}\lambda \neq 0$,

$$\begin{aligned} \lambda\Delta^{-1}(\lambda) &= \int_{-\infty}^{\infty} e^{-i\operatorname{Im}\lambda t} F(\lambda, t)\, dt \\ &= -\int_{-\infty}^{\infty} e^{-i\operatorname{Im}\lambda(t+(\pi/\operatorname{Im}\lambda))} F(\lambda, t)\, dt \\ &= -\int_{-\infty}^{\infty} e^{-i\operatorname{Im}\lambda t} F\left(\lambda, t - \frac{\pi}{\operatorname{Im}\lambda}\right) dt. \end{aligned}$$

Hence, for $\operatorname{Re}\lambda > \omega$, $\operatorname{Im}\lambda \neq 0$,

$$\left\|(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\Delta^{-1}(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\right\| \leq \frac{1}{2}\int_{-\infty}^{\infty} \left\|F\left(\lambda, t - \frac{\pi}{\operatorname{Im}\lambda}\right) - F(\lambda, t)\right\| dt.$$

From the norm continuity of $S'(t)$ for $t > 0$, it follows that $F(\lambda, t)$ is norm continuous for $t > 0$ and $t < 0$ (by (3.5)), for $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda > \omega$. Thus, for every $t \in (-\infty, \infty)$ with $t \neq 0$,

$$\lim_{|\operatorname{Im}\lambda|\to\infty} \left\|F\left(\lambda, t - \frac{\pi}{\operatorname{Im}\lambda}\right) - F(\lambda, t)\right\| = 0.$$

On the other hand, according to (3.1) and (3.5), we get

$$\|F(\lambda, t)\| \leq Ce^{-(\operatorname{Re}\lambda - \omega)|t|}, \qquad t \in (-\infty, \infty).$$

So, referring to the Lebesque dominated convergence theorem yields that

$$\lim_{|\operatorname{Im}\lambda|\to\infty} \left\|(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\,\Delta^{-1}\,(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\right\| = 0, \qquad \operatorname{Re}\lambda > \omega. \tag{3.6}$$

Using (3.1) and (3.2) together yields that for $\operatorname{Re}\lambda > \omega$, $\Delta^{-1}(\lambda) \in L(E)$ and

$$\left\|(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\Delta^{-1}(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\right\| \leq \frac{C}{\operatorname{Re}\lambda - \omega}. \tag{3.7}$$

Hence, there is $\omega_1 > \omega$ such that for $\operatorname{Re}\lambda \geq \omega_1$, $\operatorname{Im}\lambda \in (-\infty, \infty)$,

$$\left\|B(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\Delta^{-1}(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\right\| \leq \frac{1}{2},$$

since $B \in L(E)$. Thus, for $\operatorname{Re}\lambda \geq \omega_1$, $\operatorname{Im}\lambda \in (-\infty, \infty)$,

$$\begin{aligned} &\left((\operatorname{Re}\lambda + i\operatorname{Im}\lambda)^2 + A\right)^{-1} \\ &\quad = \Delta^{-1}(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\left(I - B(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\Delta^{-1}(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\right)^{-1} \in L(E), \end{aligned} \tag{3.8}$$

namely,

$$\left\{\lambda^2 : \ \lambda \in \mathbf{C},\ |\operatorname{Re}\lambda| \geq \omega_1,\ \operatorname{Im}\lambda \in (-\infty, \infty)\right\} \subset \rho(-A). \tag{3.9}$$

On the other hand, from (3.6) and (3.8), we see that there is an $\omega_2 > \omega_1$ such that for $\operatorname{Re}\lambda \geq \omega_1$, $\operatorname{Im}\lambda \geq \omega_2$,

$$\lim_{|\operatorname{Im}\lambda|\to\infty} \left\|(\operatorname{Re}\lambda + i\operatorname{Im}\lambda)\left((\operatorname{Re}\lambda + i\operatorname{Im}\lambda)^2 + A\right)^{-1}\right\| = 0. \tag{3.10}$$

Furthermore, for $\mathrm{Re}\,\eta \geq \omega_1$, we have that

$$\begin{aligned}\mu^2 + A &= \mu^2 - \eta^2 + \eta^2 + A \\ &= \left(\eta^2 + A\right)\left[I + \left(\mu^2 - \eta^2\right)\left(\eta^2 + A\right)^{-1}\right] \\ &= \left(\eta^2 + A\right)\left\{I + \left[(\mu - \eta)^2 + 2\eta(\mu - \eta)\right]\left(\eta^2 + A\right)^{-1}\right\}.\end{aligned} \tag{3.11}$$

Taking

$$\eta = \omega_1 + 1 + iq, \qquad \mu = p + iq$$

($p \in (-\omega_1 - 1, \omega_1 + 1)$, $q \in (-\infty, \infty)$), we obtain from (3.10) that there exists $q_0 \geq \omega_2$ such that for $|q| \geq q_0$,

$$\left\|\left[(\mu - \eta)^2 + 2\eta(\mu - \eta)\right]\left(\eta^2 + A\right)^{-1}\right\| \leq \frac{1}{2}. \tag{3.12}$$

Accordingly, $\eta^2 \in \rho(-A)$ implies that

$$\left\{\mu^2 : \ \mu = p + iq, \ |q| \geq q_0, \ |p| \leq \omega_1 + 1\right\} \subset \rho(-A). \tag{3.13}$$

Combining (3.13) with (3.9), we know that

$$\begin{aligned}\sigma(-A) &\subset \left\{\lambda^2 : \ |\mathrm{Re}\,\lambda|, \ |\mathrm{Im}\,\lambda| \leq q_0\right\} \\ &\subset \left\{\lambda^2 : \ |\lambda| \leq \sqrt{2}q_0\right\} \\ &\subset \left\{\lambda : \ |\lambda| \leq 2q_0^2\right\}.\end{aligned} \tag{3.14}$$

Clearly, for every $\lambda \in \{\lambda : |\lambda| \geq 4q_0^2\}$, there is a $\mu \in \{\lambda : |\lambda| \geq 2q_0\}$, such that $\mu^2 = \lambda$ and $\mathrm{Re}\,\mu \geq 0$. Moreover,

(i) if $\mathrm{Re}\,\mu \geq \sqrt{2}q_0$, then from (3.7) and (3.8) it follows that

$$\left\|\left(\mu^2 + A\right)^{-1}\right\| \leq \frac{C}{|\mu|}\frac{1}{q_0 - \omega + 1}; \tag{3.15}$$

(ii) if $\mathrm{Re}\,\mu < \sqrt{2}q_0$, then $|\mathrm{Im}\,\mu| > \sqrt{2}q_0$.

Thus, (3.11), (3.12), (3.7), and (3.8) together show that there exists a constant C_1 such that

$$\begin{aligned}\left\|\left(\mu^2 + A\right)^{-1}\right\| &\leq C_1 \left\|\Delta^{-1}(\omega_1 + 1 + i\,\mathrm{Im}\,\mu)\right\| \\ &\leq \frac{CC_1}{\left((\omega_1 + 1)^2 + (\mathrm{Im}\,\mu)^2\right)^{1/2}} \\ &\leq \frac{CC_1}{\left((\omega_1 + 1)^2 + (q_0)^2\right)^{1/2}}.\end{aligned} \tag{3.16}$$

Therefore, for every $\lambda \in \mathbf{C}$ with $|\lambda| \geq 4q_0^2$,

$$\left\|(\lambda + A)^{-1}\right\| = O(|\lambda|), \qquad \text{as } |\lambda| \to \infty, \tag{3.17}$$

which implies that

$$\left\|\frac{1}{\lambda^3}(\lambda + A)^{-1}\right\| = O\left(\lambda^{-2}\right), \qquad \text{as } |\lambda| \to \infty.$$

Fix Γ, a circle centered at the origin and enclosing $\sigma(-A)$. We obtain by virtue of [23, (5.5) on p. 64] that

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda^3}(\lambda + A)^{-1}u\,d\lambda = 0, \qquad u \in E. \tag{3.18}$$

Obviously, for all $u \in D(A^3)$, we have

$$(\lambda + A)^{-1}u = \frac{1}{\lambda}u - \frac{1}{\lambda^2}Au + \frac{1}{\lambda^3}A^2u - \frac{1}{\lambda^3}(\lambda + A)^{-1}A^3u.$$

Hence, for every $u \in D(A^3)$,

$$\frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1}u\, d\lambda = u.$$

It follows from $\overline{D(A^3)} = E$ that

$$I = \frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1}\, d\lambda.$$

Using the similar arguments as in the latter part of the proof of [23, Lemma 5.2, p. 64], we conclude that $A \in L(E)$.

SUFFICIENCY. From (3.11) in [3, p. 279], we have

$$C(t)u = u - \int_0^t S(t)Au\, dt, \qquad u \in D(A).$$

Hence, by the boundedness of A, $C(t)$ is norm continuous for $t > 0$, so is $S'(t)$ recalling the arguments below (3.3). The proof is then complete.

Corollary 2.2 is an immediate consequence of Theorem 2.1.

PROOF OF COROLLARY 2.3. Letting $B = 0$ in Theorem 2.1, we have that (i) is equivalent to (iv).

By [3, Theorem 2.3, p. 169], we know that

$$T_0(t)u := \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-s^2}C(2t^{1/2}s)u\, ds, \qquad u \in E$$

is a strongly continuous semigroup. Thus, if $C(t)$ is norm continuous at $t = 0$, so is $T_0(t)$. It follows from [4, Proposition 2.5, p. 15] or [19, Theorem 1.2, p. 2] that (ii) implies (iv). The remaining part is obvious by [20,21].

PROOF OF COROLLARY 2.4. It is well known that if A generates a strongly continuous operator group $T(t)$, then A^2 generates a cosine operator function $C(\cdot)$ given by

$$C(t) = \frac{1}{2}[T(t) + T(-t)].$$

Accordingly, Corollary 2.3 and [4, Proposition 2.5, p. 15] or [19, Theorem 1.2, p. 2] lead to the desired results.

4. AN EXAMPLE

In this section, we give an example to show that if B is unbounded, then the norm continuity for $t > 0$ of $C(t)$ and $S'(t)$ does not imply the boundedness of A.

Let P be the generator of a strongly continuous operator semigroup $\{T(t)\}_{t \geq 0}$ on E which is norm continuous for $t > 0$. Let Q be a closed linear operator on E such that $D(Q) \supset D(P)$, and $(\lambda_0 - P)^{-1}Q$ has a bounded extension for some $\lambda_0 \in \rho(P)$. Consider the following second order Cauchy problem

$$\begin{aligned} u''(t) - Pu'(t) + Qu(t) = 0, &\qquad t \geq 0, \\ u(0) = u_0, &\qquad u'(0) = u_1. \end{aligned} \tag{4.1}$$

By virtue of [13, Theorem 2], (4.1) is strongly wellposed.

Obviously, for all $u \in D(A^3)$, we have

$$(\lambda + A)^{-1}u = \frac{1}{\lambda}u - \frac{1}{\lambda^2}Au + \frac{1}{\lambda^3}A^2u - \frac{1}{\lambda^3}(\lambda + A)^{-1}A^3u.$$

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